THE PROBLEM OF TURBULENT NATURAL CONVECTION AT A VERTICAL IMPERMEABLE FLAT SURFACE

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An approximate solution is presented for the turbulent natural convection which for a local coefficient of heat transfer yields a function of the form $Nu_x \sim (Gr_x Pr)^{1/3}$. The solution is in satisfactory agreement with experimental data.

The overwhelming bulk of experimental data on turbulent natural convection developing at an impermeable vertical surface leads to a relationship between the heat-transfer coefficient and a Rayleigh number of the form $Nu_x \sim Ra_x^{1/3}$, i.e., to heat transfer that is independent of the longitudinal coordinate x.

The only theoretical solution of turbulent natural convection, obtained by Eckert and Jackson [1], yields a change in the local heat transfer from the Ra_x number according to the 0.4-law:

$$Nu_x = 0.0295 Ra_x^{0.4} \frac{Pr^{7/15}}{(1+0.494 Pr^{2/3})^{0.4}}, \tag{1}$$

which leads to noticeable divergence from experiment (Fig. 1), increasing as the Ra_x number increases.

The cause of this great divergence in the theoretical solution [1] from experimental data is to be sought in the incorrect utilization of the Blasius law, derived for forced flow, for the tangential stress at a wall in problems of natural convection.

The principal difference between natural and forced convection is the significant effect on the development of flow and heat transfer in mass (lift) forces, not taken into consideration by the Blasius formula. In the general case the use of the Blasius friction law for solution of problems in turbulent natural convection is therefore not obvious.

We employed a somewhat different approach to the analysis of the transfer of heat in turbulent natural convection, developing about impermeable vertical flat surfaces. Retaining the same expression as the Blasius law (in terms of the form of notation) for the tangential stress at the wall, we assume

$$\tau_w = C \rho u_1^2 \left(\frac{\nu}{u_1 \delta}\right)^k, \tag{2}$$

where the exponent k and the constant C are found by resort to experimental data on turbulent natural convection.

Using the Reynolds analogy with a correction factor in the form of $Pr^{-2/3}$ by means of which we take into consideration the deviation of the analogy in Prandtl numbers different from 1, we write the expression for the heat flow at the wall

$$q_w = C g \rho c_p u_1 \theta_w \left(\frac{\nu}{u_1 \delta}\right)^k Pr^{-2/3}. \tag{3}$$

The distribution of temperature and velocity in the boundary layer is assumed on the basis of the "one-seventh" law:

$$\theta = \theta_w \left[1 - \left(\frac{y}{\delta}\right)^{1/7}\right], \tag{4}$$

$$u = u_1 \left(\frac{y}{\delta}\right)^{1/7} \left(1 - \frac{y}{\delta}\right)^4, \tag{5}$$

where u_1 is some unknown expressed in units of velocity. In the last expression the factor $(1 - y/\delta)$ takes into consideration the feature of natural convection that velocity at the external edge of the boundary layer is equal to zero.

Having substituted (2)–(5) into the integral relationships of momentum

$$\frac{d}{dx} \int_0^\delta u^2 dy = g \beta \int_0^\delta \theta \, dy - \frac{\tau_w}{\rho} \tag{6}$$

and energy

$$\frac{d}{dx} \int_0^\delta u \theta \, dy = \frac{q_w}{g \rho c_p}, \tag{7}$$

we obtain the ordinary differential equations relating u_1 and the thickness of the boundary layer δ:

$$\left.\begin{aligned} 0.0523 \frac{d}{dx}(u_1^2 \delta) &= 0.125 g \beta \theta_w \delta - C u_1^2 \left(\frac{\nu}{u_1 \delta}\right)^k, \\ 0.0366 \frac{d}{dx}(u_1 \delta) &= C u_1 \left(\frac{\nu}{u_1 \delta}\right)^k Pr^{-2/3}. \end{aligned}\right\} \tag{8}$$

System (8) can be solved by the substitutions

$$u_1 = C_n x^m, \tag{9}$$

$$\delta = C_\delta x^n, \tag{10}$$

which yields

$$\left.\begin{aligned} &0.0523(2m+n) C_n^2 C_\delta x^{2m+n-1} = 0.125 g \beta \theta_w C_\delta x^n - \\ &\quad - C \nu^k C_n^{2-k} C_\delta^k x^{2m-k(m+n)}, \\ &0.0366(m+n) C_n C_\delta x^{m+n-1} = \\ &\quad = C Pr^{-2/3} \nu^k C_n^{1-k} C_\delta^{-k} x^{m-k(m+n)}. \end{aligned}\right\} \tag{11}$$

For system (11) to be satisfied for any x, the exponents of x must be equal:

$$2m + n - 1 = n = 2m - k(m+n), \tag{12}$$

$$m + n - 1 = m - k(m+n). \tag{13}$$

To find the exponents m, n, and k we thus have two conditions, (12) and (13). As the third condition we

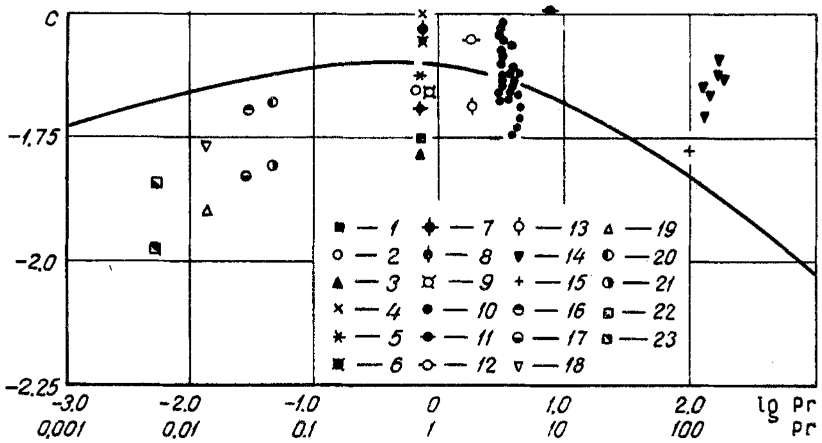

Fig. 1. Comparison of experimental data on turbulent natural convection at impermeable vertical surfaces with the values predicted by Eckert and Jackson's theory [1]($C = \lg (Nu_x/Ra_x^{0.4})$). Experiments with air: 1) Eckert and Diaguila [2]; 2) ($Ra_x = 10^{10}$) and 3) ($Ra_x = 10^{12}$) Saunders [3]; 4) ($Ra_x = 10^{10}$) and 5) ($Ra_x = 10^{12}$) Eigenson [4]; 6) ($Ra_x = 10^{10}$) and 7) ($Ra_x = 10^{12}$), King [5]; 8) ($Ra_x = 10^{10}$) and 9) ($Ra_x = 10^{12}$) Kirpichev and Gukhman [6]. Experiments with water; 10) Pchelkin [7], Pr = 3–4.5; 11) Saunders [3], Pr = 7.4; 12) ($Ra_x = 10^{10}$) and 13) ($Ra_x = 10^{12}$): Jakob and Linke [8], Pr = 1.75. Experiments with viscous liquids: 14) Pchelkin [7], transformer oil, Pr = 132–167; 15) Touloukian et al. [9] ethylene glycol, Pr = 100. Experiments by Fedynskii [10] with liquid metals on horizontal tubes; 16) $Ra = 10^{10}$ and 17) $Ra = 10^{12}$, mercury, Pr = 0.03; 18) $Ra = 10^{10}$ and 19) $Ra = 10^{12}$, tin, Pr = 0.014; 20) $Ra = 10^{10}$ and 21) $Ra = 10^{12}$, lead bismuth alloy, Pr = 0.045; 22) $Ra = 10^{10}$ and 23) $Ra = 10^{12}$, sodium, Pr = 0.005. Solid line: formula (1).

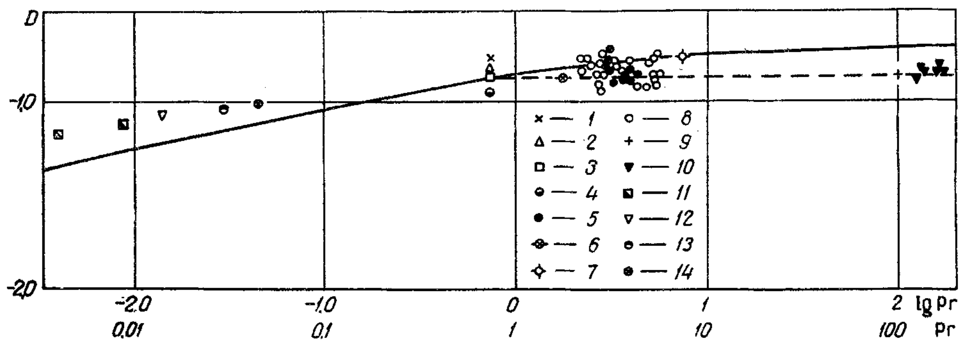

Fig. 2. Comparison of formula (24) with experimental data on turbulent convection heat transfer on impermeable vertical surfaces ($D = \lg (Nu_x/Ra_x^{1/3})$). Experiments with air (Pr = 0.72–0.73): 1) Eigenson [4]; 2) Kirpichev and Gukhman [6]; 3) King [5]; 4) Saunders [3]. Experiments with water: 5) Pchelkin [7], Pr = 3–4.5; 6) Jakob and Linke [8], Pr = 1.75 (boiling water); 7) Saunders [3], Pr = 7.4; 8) Min Kuei-jung [12]. Pr = 2.5–6. Experiments with viscous liquids: 9) Touloukian et al. [9], ethylene glycol, Pr = 100; 10) Pchelkin, transformer oil, Pr = 125–166. Experiments by Fedynskii [10] with liquid metals (Pr = 0.005–0.045) ion horizontal tubes: 11) sodium; 12) tin; 13 mercury; 14) lead -bismuth alloy. Solid line shows values predicted by formula (24), dashed line is Mikheev's curve [11], generalizing for Pr > 0.7.

may have that well-known experimental fact [2–11] that in the case of turbulent natural convection at isothermal vertical surfaces the flow of heat q_w at a wall is constant and independent of x. With substitution of (9) and (10) into (3) we have the condition

$$m - k(m+n) = 0. \quad (14)$$

The joint solution of (12)–(14) yields

$$m = n = k = 1/2. \quad (15)$$

Having introduced the values of m, n, and k into (11) and solving the resulting equations relative to the parametric constants C_n and C_δ, we have

$$C_n = 1.845\nu\,\mathrm{Pr}^{-5/6}\left(\frac{g\beta\theta_w}{\nu^2}\,\mathrm{Pr}\right)^{1/2}\times \times\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/2}, \quad (16)$$

$$C_\delta = 7.4\,C^{2/3}\left(\frac{g\beta\theta_w}{\nu^2}\,\mathrm{Pr}\right)^{-1/6}\times \times\left(\frac{2.14+\mathrm{Pr}^{2/3}}{\mathrm{Pr}^{2/3}}\right)^{1/6}\mathrm{Pr}^{-1/6}, \quad (17)$$

whence with consideration of (9) and (10) we find the thickness of the boundary layer and the quantity u_1:

$$\delta = 7.4C^{2/3}\left(\frac{g\beta\theta_w}{\nu^2}\,\mathrm{Pr}\right)^{-1/6}\times \times\left(\frac{2.14+\mathrm{Pr}^{2/3}}{\mathrm{Pr}^{2/3}}\right)\mathrm{Pr}^{-1/6}x^{1/2}, \quad (18)$$

$$u_1 = 1.845\nu\,\mathrm{Pr}^{-5/6}\left(\frac{g\beta\theta_w}{\nu^2}\,\mathrm{Pr}\right)^{1/2}\times \times\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/2}x^{1/2}. \quad (19)$$

The relative boundary-layer thickness δ/x and the Reynolds number constructed from the maximum velocity of the boundary layer $\mathrm{Re}_{max} = u_{max}x/\nu$, where $u_{max} = 0.537u_1$, are defined by the formulas

$$\frac{\delta}{x} = 7.4C^{2/3}\,\mathrm{Ra}_x^{-1/6}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{-1/6}\mathrm{Pr}^{-1/6}, \quad (20)$$

$$\mathrm{Re}_{max} = 0.99\mathrm{Ra}_x^{1/2}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/2}\mathrm{Pr}^{-5/6}. \quad (21)$$

Having substituted the heat flow q_w from (3) into the local Nusselt number $\mathrm{Nu}_x = q_w x/\theta_w\lambda$, with consideration of (18) and (19), we obtain

$$\mathrm{Nu}_x = \frac{1}{2}\,C^{2/3}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/3}\mathrm{Ra}_x^{1/3}. \quad (22)$$

The greatest number of experiments on turbulent natural convection has been carried out with air, and for the complex

$$\xi = \frac{1}{2}\,C^{2/3}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/3} \quad (23)$$

these yield the following values: $\xi = 0.135$ (after Mikheev [11]); 0.140 (after Kirpichev and Gukhman [6]); 0.148 (after Eigenson [4]); 0.109 (after Saunders [3]); and 0.130 (after King [5]).

Assuming an average value of $\xi = 0.13$ for $\mathrm{Pr} = 0.72$, from (23) we find the constant $C = 0.253$. If we introduce this into (22), (20), (2), and (3), we will finally obtain

$$\mathrm{Nu}_x = 0.2\mathrm{Ra}_x^{1/3}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{1/3}, \quad (24)$$

$$\frac{\delta}{x} = 2.96\mathrm{Ra}_x^{-1/6}\left(\frac{\mathrm{Pr}^{2/3}}{2.14+\mathrm{Pr}^{2/3}}\right)^{-1/6}\mathrm{Pr}^{-1/6}, \quad (25)$$

$$\tau_w = 0.253\rho u_1^2\left(\frac{\nu}{u_1\delta}\right)^{1/2}, \quad (26)$$

$$q_w = 0.253g\rho c_p\theta_w u_1\left(\frac{\nu}{u_1\delta}\right)^{1/2}\mathrm{Pr}^{-2/3}. \quad (27)$$

In Fig. 2 we have a comparison of the experimental data with formula (24). As we can see from the figure, formula (24) shows considerably better convergence with experiment than the Eckert and Jackson solution (see Fig. 1).

In view of the absence of experimental data on turbulent natural convection on vertical surfaces in the region of $\mathrm{Pr} \ll 1$ the experimental Fedynskii [10] data on the average transfer of heat in liquid metals in the case of turbulent natural convection in horizontal tubes are plotted in rough approximation in Figs. 1 and 2. As was to be expected, the Fedynskii experiments lie somewhat higher than the theoretical solution, retaining the fundamental relationship of heat transfer as a function of the Pr number expressed by formula (24).

NOTATION

x and y are longitudinal and transverse coordinates; Nu_x is the local Nusselt number; $\mathrm{Ra}_x = \mathrm{Gr}_x\mathrm{Pr}$ is the local Rayleigh number; τ_w and q_w are the shear stress and heat flux at the wall; u_1 is the characteristic velocity for natural convection; δ is the boundary layer thickness; θ is the difference between the boundary layer and free stream; θ_w is the temperature difference between the wall and the free stream; Ra is the mean Rayleigh number.

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